Intro to Matrices

Don't be scared…

What is a matrix?

- A Matrix is just rectangular arrays of items
- A typical matrix is a rectangular array of numbers arranged in rows and columns.

$$\underset{3x4}{A} = \begin{bmatrix} 21 & 62 & 33 & 93 \\ 44 & 95 & 66 & 13 \\ 77 & 38 & 79 & 33 \end{bmatrix}$$

Sizing a matrix

- By convention matrices are "sized" using the number of rows (m) by number of columns (n).

$$\underset{3x4}{A} = \begin{bmatrix} 21 & 62 & 33 & 93 \\ 44 & 95 & 66 & 13 \\ 77 & 38 & 79 & 33 \end{bmatrix} \quad \underset{3x3}{B} = \begin{bmatrix} 7 & 3 & 2 \\ 8 & 4 & 1 \\ 6 & 5 & 9 \end{bmatrix}$$

$$\underset{4x2}{C} = \begin{bmatrix} 11 & 4 \\ 14 & 7 \\ 16 & 8 \\ 22 & 3 \end{bmatrix} \quad \underset{1x1}{D} = \begin{bmatrix} 17 \end{bmatrix}$$

"Special" Matrices

- Square matrix: a square matrix is an mxn matrix in which m = n.

$$\underset{3x3}{B} = \begin{bmatrix} 7 & 3 & 2 \\ 8 & 4 & 1 \\ 6 & 5 & 9 \end{bmatrix}$$

- Vector: a vector is an mxn matrix where either m OR n = 1 (but not both).

$$\underset{4x1}{X} = \begin{bmatrix} 12 \\ 9 \\ -4 \\ 0 \end{bmatrix} \qquad \underset{1x3}{Y} = \begin{bmatrix} 7 & -22 & 14 \end{bmatrix}$$

"Special" Matrices

- Scalar: a scalar is an mxn matrix where BOTH m and n = 1.

$$\underset{1x1}{D} = \begin{bmatrix} 17 \end{bmatrix}$$

- Zero matrix: an mxn matrix of zeros.

$$\underset{3x2}{0} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

- Identity Matrix: a square (mxm) matrix with 1s on the diagonal and zeros everywhere else.

$$\underset{3x3}{I} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Matrix Rank

- Matrix Rank: the rank of a matrix is the maximum number of linearly independent vectors (either row or column) in a matrix
- Full Rank: A matrix is considered full rank when all vectors are linearly independent

Transposing a Matrix

- Matrix Transpose: is the mxn matrix obtained by interchanging the rows and columns of a matrix (converting it to an nxm matrix)

$$\underset{4x1}{X} = \begin{bmatrix} 12 \\ 9 \\ -4 \\ 0 \end{bmatrix} \quad \underset{1x4}{X'} = \begin{bmatrix} 12 & 9 & -4 & 0 \end{bmatrix}$$

$$\underset{3x4}{A} = \begin{bmatrix} 21 & 62 & 33 & 93 \\ 44 & 95 & 66 & 13 \\ 77 & 38 & 79 & 33 \end{bmatrix} \quad \underset{4x3}{A'} = \begin{bmatrix} 21 & 44 & 77 \\ 62 & 95 & 38 \\ 33 & 66 & 79 \\ 93 & 13 & 33 \end{bmatrix}$$

Matrix Addition

- Matrices can be added (or subtracted) as long as the 2 matrices are the same size
  - Simply add or subtract the corresponding components of each matrix.

$$\underset{2x3}{A} = \begin{bmatrix} 1 & 2 & 3 \\ 7 & 8 & 9 \end{bmatrix} \quad \underset{2x3}{B} = \begin{bmatrix} 5 & 6 & 7 \\ 3 & 4 & 5 \end{bmatrix}$$

$$A+B = \begin{bmatrix} 1 & 2 & 3 \\ 7 & 8 & 9 \end{bmatrix} + \begin{bmatrix} 5 & 6 & 7 \\ 3 & 4 & 5 \end{bmatrix} = \begin{bmatrix} 1+5 & 2+6 & 3+7 \\ 7+3 & 8+4 & 9+5 \end{bmatrix} = \begin{bmatrix} 6 & 8 & 10 \\ 10 & 12 & 14 \end{bmatrix}$$

$$A+B = B+A$$

$$A-B = \begin{bmatrix} 1 & 2 & 3 \\ 7 & 8 & 9 \end{bmatrix} - \begin{bmatrix} 5 & 6 & 7 \\ 3 & 4 & 5 \end{bmatrix} = \begin{bmatrix} 1-5 & 2-6 & 3-7 \\ 7-3 & 8-4 & 9-5 \end{bmatrix} = \begin{bmatrix} -4 & -4 & -4 \\ 4 & 4 & 4 \end{bmatrix}$$

Matrix Multiplication

- Multiplying a matrix by a scalar: each element in the matrix is multiplied by the scalar.

$$\underset{2x3}{A} = \begin{bmatrix} 1 & 2 & 3 \\ 7 & 8 & 9 \end{bmatrix} \text{ and } \underset{1x1}{x} = 5\text{; then}$$

$$xA = \begin{bmatrix} 5*1 & 5*2 & 5*3 \\ 5*7 & 5*8 & 5*9 \end{bmatrix} = \begin{bmatrix} 5 & 10 & 15 \\ 35 & 40 & 45 \end{bmatrix}$$

Matrix Multiplication

- Multiplying a matrix by a matrix:
  - the product of matrices A and B (AB) is defined if the number of columns in A equals the number of rows in B.
  - Assuming A has ixj dimensions and B has jxk dimensions, the resulting matrix, C, will have dimensions ixk
  - In other words, in order to multiply them the inner dimensions must match and the result is the outer dimensions.
  - Each element in C can by computed by:

$$C_{ik} = \Sigma_j A_{ij} B_{jk}$$

Matrix Multiplication

- Multiplying a matrix by a matrix:

$$\underset{2x3}{A} = \begin{bmatrix} 1 & 2 & 3 \\ 7 & 8 & 9 \end{bmatrix} \quad \underset{3x2}{B'} = \begin{bmatrix} 5 & 3 \\ 6 & 4 \\ 7 & 5 \end{bmatrix}$$

$$\underset{2x3}{A}\,\underset{3x2}{B'} = \underset{2x2}{C} = \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix}$$

Matching inner dimensions!!
Resulting matrix has outer dimensions!!!

Reducing Square Matrices

- Trace: the sum of the diagonal of a square matrix.

$$\underset{3x3}{B} = \begin{bmatrix} 7 & 3 & 2 \\ 8 & 4 & 1 \\ 6 & 5 & 9 \end{bmatrix}$$

$$tr(B) = 7 + 4 + 9 = 20$$

Reducing Square Matrices

- Determinant:
  - The determinant of a matrix is a scalar representation of matrix; considered the “volume” of the matrix or in the case of a VCV matrix it is the generalized variance.
  - Only square matrices have determinants.
  - Determinants are also useful because they tell us whether or not a matrix can be inverted (next).
  - Not all square matrices can be inverted (must be full rank, non-singular matrix)

Reducing Square Matrices

- Determinant:

$$\underset{1x1}{C} = [4] \qquad |C| = 4$$

$$\underset{2x2}{C} = \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix} \qquad |C| = (a_1 * b_2) - (b_1 * a_2)$$

$$\underset{2x2}{C} = \begin{bmatrix} 3 & 2 \\ 5 & 1 \end{bmatrix} \qquad |C| = (3*1) - (2*5) = 3 - 10 = -7$$

Reducing Square Matrices

- Determinant:

$$\underset{3x3}{C} = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix} \qquad |C| = a_1 \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - b_1 \begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} + c_1 \begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}$$

$$\underset{3x3}{C} = \begin{bmatrix} 2 & -2 & 0 \\ -1 & 5 & 1 \\ 3 & 4 & 5 \end{bmatrix}$$

$$|C| = [2(5*5-1*4)] - [-2(-1*5-1*3)] + [0(-1*5-5*3]$$

$$|C| = [2(25-5)] - [-2(-5-3)] + [0(-5-15]$$

$$|C| = [40] - [16] + [0] = 24$$

Matrix Inverse

- Matrix Inverse: Needed to perform the "division" of 2 square matrices
  - In scalar terms A/B is the same as A * 1/B
  - When we want to divide matrix A by matrix B we simply multiply by A by the inverse of B
  - An inverse matrix is defined as

$$\underset{nxn}{A^{-1}} \xrightarrow{Defined} \underset{nxn}{A}\,\underset{nxn}{A^{-1}} = \underset{nxn}{I} \;\; AND \;\; \underset{nxn}{A^{-1}}\,\underset{nxn}{A} = \underset{nxn}{I}$$

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Matrix Inverse

- Matrix Inverse:
  - For a 2x2 matrix the inverse is relatively simple

$$\underset{2x2}{C} = \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix} \qquad \underset{2x2}{C^{-1}} = \frac{1}{|C|}\begin{bmatrix} a_1 & -b_1 \\ -a_2 & b_2 \end{bmatrix}$$

$$\underset{2x2}{C} = \begin{bmatrix} 3 & 2 \\ 5 & 1 \end{bmatrix} \qquad |C| = -7$$

$$\underset{2x2}{C^{-1}} = \frac{1}{-7}\begin{bmatrix} 3 & -2 \\ -5 & 1 \end{bmatrix} = \begin{bmatrix} -3/7 & 2/7 \\ 5/7 & -1/7 \end{bmatrix}$$

  - For anything else, use a computer…

Singular Matrix

- Singular Matrix: A matrix is considered singular if the determinant of the matrix is zero
  - The matrix cannot be inverted
  - Usually caused by linear dependencies between vectors
  - When a matrix is not full rank

$$\underset{2x2}{A} = \begin{bmatrix} 2 & 6 \\ 1 & 3 \end{bmatrix} \qquad |A| = (2*3) - (1*6) = 0$$

  - An extreme form of multicollinearity in the matrix